A Class of Exact Solutions for the Vlasov-Turbulence

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(Z. Naturforsch. **24 a**, 711—715 [1969]; received 21 February 1969)

The admissible closure conditions can be completely determined in the limit of vanishing correlation length. In this case the Lundgren hierarchy can be solved, since this irregular state persists in the course of time. The electric field on the other hand loses its statistical character. — The results can be extended to examples of homogeneous turbulence with a finite correlation length.

In a preceding paper [1] we deduced a hierarchy of equations for the statistical description of Vlasov-turbulence in terms of its joint probability distributions. This corresponds very closely to a result which has been given in the case of ordinary Navier-Stokes-turbulence by LUNDGREN [2]. We were able to define, at least in principle, the admissibility of possible closure conditions for the infinite set of equations.

For the case of a product measure a complete treatment is posible. As was to be expected, even in this restricted case a wide variety of solutions can be found. Since the corresponding stochastic field is highly irregular, it would be better to consider this as some kind of limiting case for very short correlation lengths. In this sense one can expect the integral-expressions for the electric fields to exist. In particular it follows that the electric field becomes statistically sharply distributed. This fact allows a simplified derivation of the time-dependence of the distribution-functions.

The first chapter is devoted to these derivations. In the second chapter we consider some possible methods of solving the equation of motion. The results can be generalized by superposing different statistical ensembles. This procedure is comparable to the introduction of the density matrix in quantum-mechanics. In particular examples of homogeneous turbulence can be achieved this way. The restriction to one dimension is only for convenience.

1. A Class of Exact Solutions

It seems laborious to establish the restriction on P_1 for the admissibility of higher orders. But there is a special case, for which the argument of (I) can be extended to all orders. The simplest case corresponds to a product:

$$P_k(1,2,\dots,k) = P_1(1)\,P_1(2)\dots P_1(k) \qquad (1)$$

if all (x_i, u_i) are different. Clearly all the Kolmogorov requirements are fulfilled. Inserting (1) into

$$\frac{\partial P_k}{\partial t} + \sum_{j=1}^{k} \Big\{ u_j \frac{\partial P_k}{\partial x_j} + \int G(x_j - x_{k+1})\,\mathrm{d}x_{k+1}\,\mathrm{d}u_{k+1}\,\frac{\partial}{\partial u} \cdot \int P_{k+1} f_{k+1}\,\mathrm{d}f_{k+1} = 0$$

yields the following sufficient relation:

$$\frac{\partial P_1}{\partial t} + u\,\frac{\partial P_1}{\partial x} + E\,\frac{\partial P_1}{\partial u} = 0 \qquad (2)$$

with

$$\frac{\partial E}{\partial x} = \int \mathrm{d}u \int \mathrm{d}f\, f\, P_1 - 1 \qquad (3)$$

if we return to the Vlasov case with fixed ion-background.

From (3) we see, that $E(t,x)$ is the mean value of the electric field. If we calculate its fluctuation, we get upon using the correlationless assumption

$$\overline{E^2}(t,x) = \iint G(x-x')\,G(x-x'') \cdot \bar f(x',u',t)\,\bar f(t,x'',u'')\,\mathrm{d}x'\,\mathrm{d}u'\,\mathrm{d}x''\,\mathrm{d}u''.$$

Hence this is $\quad = \bar E^2(t,x)$

which means there is no fluctuation within the electric field:

$$P(t,x\colon E) = \delta(E - E(t,x)).$$

Despite the fact, that $f(t,x,u)$ is itself very randomly distributed one obtains for such integral expressions of f as the electric field a sharp — non-random — distribution. We want to discuss the reason

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[1] P. GRÄFF, Z. Naturforsch. **24 a**, 701 [1969]; in the following cited as (I).
[2] T. S. LUNDGREN, Phys. Fluids **10**, 969 [1967].

for this behaviour which also will lead us to a more direct derivation of (2) and (3) without using the Lundgren-hierarchy.

The ansatz (1), corresponding to uncorrelated values in immediately neighbouring points, evidently leads to a set of very irregular and discontinuous sample functions.

Hence one may wonder how an integral – such as that needed for calculating $E(x)$ – could even exist. Actually, due to a theorem of DOOB[3], this does not hold. Nevertheless, we may consider our case as a limiting situation of more regular ones, which are described by a short-range correlation length l, if we let the latter tend to zero. Actually, in this limit, the integrals are independent from the way in which we let $l \to 0$ and also the influence of the correlation-terms remains small for any finite time.

The following remarks offer no exact proof, but only suggest, why this may be possible. We consider at first a game, which yields a single sample function in accordance with the statistics (1). For simplicity we suppress the variable u and restrict ourselves to a unit interval $0 \leqq x \leqq 1$ assuming further, that $P_1(x: f)$ should not depend on x. Then we have for different points x_i:

$$P_k(x_1: f_1; \ldots; x_k: f_k) = \prod_{i=1}^{k} P_1(f_i).$$

One may be tempted to construct a sample function by the following play. Choose f_1 at point $x_1 = 0$ from a sequence of random numbers whose magnitude are ruled by $P_1(f)$, then at point $x_2 = 1$ choose f_2 in the same fashion and so on at $x_3 = 1/2$ etc. What we get is a denumerable sequence of points (x_i, f_i).

Define

$$f^{(n)}(x) = f_i \quad \text{if} \quad x_i \leqq x < x_{i+1}, \qquad i \leqq n$$

which is a sequence of step-functions. For any two arbitrary points $x \neq y$ we can always find an index $N(x, y)$, such that $f^{(n)}(x)$ is distributed independently from $f(y)$ for $n > N(x, y)$. In this sence $f^{(n)} \to f$.

Now consider, for example, the integral

$$I = \int_0^1 f(x)\, \mathrm{d}x$$

which we approximate by

$$I_n = \int_0^1 f^{(n)}(x)\, \mathrm{d}x.$$

In the sense of Lebesgue, we must ask for the measure of the set of all points for which

$$f(x) \leqq \alpha.$$

Evidently since f was obtained from an infinite sequence of random numbers ruled by $P(f)$, the above mentioned set of points should be measurable and its measure given by

$$\text{Prob}\,\{f(x) \leqq \alpha\}. \tag{4}$$

Especially if we consider the $f^{(n)}$, this is evidently converging to (4). Hence we expect

$$I = \int f\, P(f)\, \mathrm{d}f$$

which shows that we get this "ergodic" result for any of the sample functions. This corresponds to the fact, that I is sharply distributed:

$$P(I) = \delta(I - \bar{f}).$$

Now, if we had chosen some other sequence which also forms a dense set, we would have obtained the same result. In fact, writing I_n explicitly as

$$I_n = \sum_{i=1}^{n} f_i^{(n)} \cdot \Delta x_i^{(n)}$$

where $\quad \Delta x_i^{(n)} = x_i^{(n)} - x_{i-1}^{(n)}$

we get

$$\mathcal{E}\{(I_n - \mathcal{E}\{I_n\})^2\} = \sum_i [\mathcal{E}\{f_i^{(n)2}\} - (\mathcal{E}\{f_i^{(n)}\})^2]\, \Delta^2 x_i^{(n)}$$
$$\leqq l_n\,(\mathcal{E}(f^2) - (\mathcal{E}(f))^2).$$

Here l_n is defined as

$$l_n = \max(\Delta x_i^{(n)})$$

which tends to zero for $n \to \infty$, and hence our integrals converge to a fixed sharply distributed value. In this sense, the limit is independent from the way in which it is approached.

Also within the equations of motion, we expect small corrections, that is,

$$\frac{\partial P_1}{\partial t} + u \frac{\partial P_1}{\partial x} + E \frac{\partial P_1}{\partial u} + O(l_n) = 0.$$

In the considered limit the influence of the correction terms may be made as small as desireable for any finite time-interval.

[3] J. L. DOOB, Trans. Am. Math. Soc. **42**, 107 [1937], Theorem 2.4. Compare also his textbook: Stochastic processes, Wiley, New York 1953, chapt. III.

Also we remark, that any Steklov-averaged function of a sample

$$f_\varepsilon(x) = \frac{1}{2\,\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} f(x')\,\mathrm{d}x'$$

should be sharply distributed, and in case of rather continuous averages f we expect for any small $\varepsilon > 2\,\delta_n$

$$f_\varepsilon(x) \approx \bar{f}(x)$$

which shows, that we are treating an extraordinary small scale turbulence. Due to this behaviour we expect similar results to hold for $E(t, x)$. But if $E(t, x)$ is sharply distributed, the characteristic mapping of (I. 8) would apply. Expansion of this formula (I. 8) leads back to (2).

2. Solutions

Our equation for P_1 corresponds very nearly to the original Vlasov equation. In fact, if we multiply (2) by f and integrate, we get

$$\frac{\partial \bar{f}}{\partial t} + u\,\frac{\partial \bar{f}}{\partial x} + E\,\frac{\partial \bar{f}}{\partial u} = 0 \tag{5}$$

whereas Poissons equation (3) reads

$$\frac{\partial E}{\partial x} = \int \mathrm{d}u\,\bar{f} - 1\,. \tag{6}$$

This shows that the mean-value of f has to be a self-consistent solution of Vlasovs equation. If we assume the system (5) and (6) to be solved by some function $\bar{f}(t, x, u)$, we may easily construct solutions of Eq. (2) for P_1. A simple possibility is the following:

Let $p(\xi)$ be a distribution function in one variable with

$$p(\xi) = 0 \quad \text{if} \quad \xi < 0 \tag{7}$$

and

$$\int p(\xi)\,\xi\,\mathrm{d}\xi = 1\,. \tag{8}$$

Then we define

$$P_1(t, x, u\colon f) = \begin{cases} p\left(\dfrac{f}{\bar{f}(t,x,u)}\right)\dfrac{1}{\bar{f}(t,x,u)} & \text{if } \bar{f} \neq 0\,,\\ \delta(f) & \text{if } \bar{f}(t,x,u) = 0\,. \end{cases} \tag{9}$$

It is evident, that both possibilities for P_1 are continuously connected, if $\bar{f}(t, x, u)$ is itself a continuous function. Therefore we may consider P_1 as a function of $\bar{f}(t, x, u)$:

$$P_1(t, x, u\colon f) = \text{function}\,(\bar{f}(t, x, u))$$

where f only plays the role of a parameter. But since $\bar{f}$ is a solution of (5) – or (2) – any function of $\bar{f}$ is also a solution of (5) or (2). Hence P_1 is a solution of (2). It is further evident that (3) is fulfilled, since we have by (7) and (8)

$$\int P_1(t, x, u\colon f)\,f\,\mathrm{d}f = \bar{f}\,(t, x, u) \tag{10}$$

and therefore our P_1 produces just the correct self-consistent electric field, because $\bar{f}$ was assumed to be a selfconsistent solution.

It is remarkable, that with the help of any self-consistent solution $\bar{f}(t, x, u)$ of the original Vlasov-equation we are able to construct a wide variety of possible solutions of the Lundgren-hierarchy. The arbitrariness enters through the distribution function $p(\xi)$, which is only very weakly restricted by (7) and (8). Now we will show, that an even larger class of arbitrary funcitons is consistent with Eqs. (2) and (3). To this end let us assume $\bar{f}(t, x, u)$ to be bounded

$$\bar{f}(t, x, u) < C\,.$$

Then for any constant α with

$$0 < \alpha < 1/C \tag{11}$$

we have $\quad \bar{f} - \alpha\bar{f}^2 > 0 \quad \text{if} \quad \bar{f} > 0$

Now consider $n+1$ arbitrary functions p_k, all under restriction (7) and (8), and choose arbitrary positive constants β_i with

$$\sum_{i=1}^{n+1} \beta_i = 1\,.$$

Then $P_1(t, x, u\colon f)$:

$$\begin{aligned} &= \sum_1^n \beta_k^{\,2}\,p_k\left(\frac{\beta_k\,f}{\alpha^{k-1}(\bar{f}^k - \alpha\,\bar{f}^{k+1})}\right)\frac{1}{\alpha^{k-1}(\bar{f}^k - \alpha\,\bar{f}^{k+1})}\\ &\quad + \beta_{n+1}\,p_{n+1}\left(\frac{\beta_{n+1}\,f}{\alpha^n\,\bar{f}^n}\right)\frac{1}{\alpha^n\,\bar{f}^n} \qquad \text{if } \bar{f} \neq 0\\ &= \delta(f) \qquad \text{if } \bar{f} = 0 \end{aligned}$$

is a normalized distribution and

$$\int P_1\,f\,\mathrm{d}f = \sum_1^n \alpha^{k-1}(\bar{f}^k - \alpha\bar{f}^{k+1}) + \alpha^n\bar{f}^n = \bar{f}$$

as is demanded by (10). That P_1 is a solution of (2) is evident by its construction. In particular, if we consider sequences $\{\beta_k, p_k\}$ the limit $n \to \infty$ applies, since

$$\alpha^n\,\bar{f}^n \to 0$$

by (11).

The general idea behind these constructions is the following: Assume the electric field $E(t,x)$ as a given parameter. Consider a sequence $\{\varphi_k\}$ of different solutions of

$$\frac{\partial \varphi_k}{\partial t} + u\frac{\partial \varphi_k}{\partial x} + E\frac{\partial \varphi_k}{\partial u} = 0$$

such that

$$\frac{\partial E}{\partial x} + 1 = \sum_k \int \varphi_k(t,x,u)\, du\,.$$

Then it is posible to build up with the help of distributions whose mean-value is 1, a common distribution

$$\sum_{k=1}^{\infty} \beta_k^2\, p_k\left(\frac{\beta_k f}{\varphi_k(t,x,u)}\right)\frac{1}{\varphi_k(t,x,u)}$$

which is a solution of (2) and (3) as before. Surely the above considerations will not exhaust all possible solutions. But they do demonstrate that there exists a manifold of solutions, which are consistent with a given mean electric field, and which are much more numerous than was expected from the original Vlasov equation.

This enlarged range of possibilities for f enters into the description of all nonlinear expectation values of f such as the entropy

$$-\mathcal{E}\{\int f \ln f\, dx\, du\}$$

whereas e. g. the mean temperature or

$$\mathcal{E}\{\int f u^2\, du\}$$

and similar expressions are not influenced by the different choices of P_1.

There are some far reaching differences between the ordinary solution of the Vlasov equation and our statistical treatment.

(i) Despite the fact, that we may assume the mean value $\bar{f}(t\,x,u)$ to be a continuous function, our product measure (1) enforces the corresponding sample-functions to be highly discontinuous.

(ii) We are not able for instance to describe a homogeneous turbulent plasma. We have of course:

$$\bar{E}(t,x) = E(t,x) = 0\,.$$

But since E was sharply distributed E would vanish identically.

We may overcome these difficulties at least partially by using te superposition properties (I, the end of chapt. 1). Choose positive constants γ_ϱ with

$$\sum_\varrho \gamma_\varrho = 1$$

and assume different solutions of the previously discussed form to be given: Then we may mix up a new solution by

$$P_k(1,\ldots,k) = \sum_\varrho \gamma_\varrho P_1^\varrho(1)\ldots P_1^\varrho(k) \qquad (12)$$

where the P_1^ϱ are determined as solutions of

$$\frac{\partial P_1^\varrho}{\partial t} + u\frac{\partial P_1^\varrho}{\partial x} + E^\varrho\frac{\partial P_1^\varrho}{\partial u} = 0$$

and

$$\frac{\partial E^\varrho}{\partial x} + 1 = \int du\, df\, P_1^\varrho.$$

Now we get for the average electric field

$$\bar{E}(t,x) = \sum_\varrho \gamma_\varrho E^\varrho(t,x) \qquad (13)$$

and for its correlation

$$\overline{E(t,x)\cdot E(t,y)} = \sum_\varrho \gamma_\varrho E^\varrho(t,x)\, E^\varrho(t,y)\,. \qquad (14)$$

This is evidently

$$\neq \bar{E}(t,x)\, \bar{E}(t,y)$$

and hence we have a finite correlation.

It should be mentioned, that (13) and (14) yield in some sense a possible answer to the question of the closure of the electric field-equations: They describe a loose connection between the average electric field and its correlation, which may be formulated in the form of the following representation theorem: An average electric field $\bar{E}(x,t)$ is compatible with an average field-correlation $\varrho(x,y)$, if there exist constants $\gamma_1, \gamma_2, \ldots$ and solutions of Vlasovs equation $f^1, f^2, \ldots$ with corresponding selfconsistent electric fields $E^1, E^2, \ldots$ in such a way, that the following representations hold:

$$\bar{E}(t,x) = \sum \gamma_\varrho E^\varrho(t,x)\,,$$
$$\varrho(x,y,t) = \sum \gamma_\varrho E^\varrho(t,x)\, E^\varrho(t,y)\,.$$

But of course, these are only sufficient conditions for admissible E-fields, due to the special ansatz (12) which we have chosen.

The result which we have just derived, is not yet well suited for the description of homogeneous turbulence. We may easily obtain $\bar{E}=0$, for instance with

$$\gamma_1 = \gamma_2 = 1/2 \qquad E^1(t,x) = -E^2(t,x)$$

and simultaneously $\varrho \neq 0$. But we want two further properties of ϱ:

ϱ should only depend on $|x-y|$, and

$$\lim_{|x-y|\to\infty} \varrho(x,y) = 0$$

where the last property describes some kind of clustering (I [10]).

In this situation one should apply a different mixing procedure such as is used in (I. 10) and consider the limit $L \to \infty$

$$\bar{P}_1(t,x,u:f) = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{+L} \sum \gamma_\varrho P_1^\varrho(t,x+\xi,u:f)\,d\xi\,.$$

Corresponding formulas apply for the higher distributions. In particular we get for the mean-value of the electric field:

$$\langle \bar{E}(t,x) \rangle = \sum \gamma_\varrho \langle E^\varrho \rangle = \text{independent of } x$$

if we denote the "smearing out" by brackets. The correlation now automatically depends only on the difference $(x-y)$. The clustering can be achieved, if $E(x,t)$ has the space-dependence of a so-called "fonction pseudo-aleatoires" [4].

We have not avoided by this procedure the discontinuity of the sample functions. This can be seen as follows:

If they were continuous, we would expect continuity in the mean:

$$\mathcal{E}\{[f(t,x+h,u)-f(t,x,u)]^2\} \to 0$$

for $h \to 0$. Explicitly stated for $x=0$:

$$\frac{1}{2L}\int_{-L}^{+L} \sum \gamma_\nu \big(\overline{f_\nu^2}(t,x+h,u) + \overline{f_\nu^2}(t,x,u) - 2\bar{f}_\nu(t,x+h,u)\bar{f}_\nu(t,x,u)\big)\,dx\,.$$

Because of the homogeneity of the result, we are allowed to write $\overline{f_\nu^2}(t,x,u)$ instead of $\overline{f_\nu^2}(t,x+h,u)$ and we get

$$\frac{1}{L}\int_{-L}^{+L} \sum \gamma_\nu \{\overline{f_\nu^2}(t,x,u) - \bar{f}_\nu(t,x+h,u)\bar{f}_\nu(t,x,u)\}\,dx\,.$$

Expanding around $h=0$ yields

$$\frac{1}{L}\int_{-L}^{+L} \sum \gamma_\nu \Big(\overline{f_\nu^2}(t,x,u) - \bar{f}_\nu^2(t,x,u) - \frac{h^2}{2}\bar{f}_\nu(t,x,u)\cdot\frac{\partial^2 \bar{f}_\nu(t,x,u)}{\partial x^2} + \dots\Big)\,dx\,.$$

In order to have this expression vanish it is necessary that

$$\lim_{L\to\infty}\frac{1}{L}\int_{-L}^{+L} \big(\overline{f_\nu^2}(t,x,u) - \bar{f}_\nu^2(t,x,u)\big)\,dx = \lim_{L\to\infty}\frac{1}{L}\int_{-L}^{+L} \mathcal{E}\{(f_\nu(t,x,u) - \bar{f}_\nu(t,x,u))^2\}\,dx$$

should vanish. But evidently this is only possible, if f_ν differs from its mean-value only in a finite volume, which is without influence in the limit $L \to \infty$. This contradicts an assumption of homogeneous turbulence.

Concluding Remarks

We have treated a special case of highly irregular Vlasov-turbulence which is represented by the limit of vanishing correlation length. It is shown that this state persists in the course of time, whereas the electric field loses its statistical character. Due to this fact, solutions of the statistical equations can be constructed from the solutions of the original Vlasov-equation.

Finally the result can be extended in such a way that examples of finite corelation-length and homogeneous turbulence can be built up.

I wish to thank Dr. D. PFIRSCH and Prof. Dr. W. v. WALDENFELS for helpful comments and discussions.

[4] J. BASS, Les fonctions pseudo-aleatoires, Gauthier-Villars, Paris 1962.